INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 7, 919–925 (1987)

AN ALGORITHM FOR THE GLOBAL VOLUME CONSERVATION CONSTRAINT IN LAKE CIRCULATION

GOURI DHATT*

Université Laval (Québec) and Université de Technologie de Compiègne (France), Compiègne 60200 Cedex, France

AND

MICHEL LECLERC*

Institut National de la Recherche Scientifique-Eau (Québec) C.P. 7500, Sainte-Foy (Québec), Canada GIV 4C7

SUMMARY

We present an algorithm for introducing a global constraint of volume conservation in lake circulation problems. The algorithm is described for linear problems, and is then generalized to non-linear cases. Numerical examples are presented to show the influence of water-level conditions on convergence and to demonstrate the practical superiority of the global constraint algorithm for obtaining reliable convergent solutions.

INTRODUCTION

The finite element modelling of hydraulic flow problems is considered reliable and efficient by practising engineers. Since the pioneering works of Taylor and Hood,[1] and Connor and Brebbia[2], various researchers have given significant contributions to make the method robust and versatile for practical applications. One may mention a recent simulation study of the water system of the Montreal archipelago employing finite elements, which has been accepted to be highly economical and rapid by the environmental engineers and biologists of Hydro-Québec.[3]

For practical applications, boundary conditions have to be properly chosen and introduced in the model. In free surface flows, it is observed that the convergence and precision are very sensitive to water-level variations. In a recent work on three-dimensional wind-induced lake circulation,[4] it has been shown that the quality and convergence of a solution are closely related to the proper definition of water-level boundary conditions. In fact, it is not possible to choose a water-level reference point, as it varies significantly with wind direction and intensity and the geometry of the domain. The most appropriate condition for recirculation is conservation of total volume, and all water level variations must be subjected to this global constraint for a realistic solution convergence.

The present model contains the horizontal velocity components $u(x, y, z)$ and $v(x, y, z)$ and the water level $h(x, y)$, all as unknowns. Since the model is stationary, one necessarily requires an

* Professor

0271–2091/87/090919–07$05.00

Received January 1987

explicit boundary condition on h to conserve volume. However, if a temporal formulation is employed, no explicit volume conservation constraint may be required since it is included in the formulation.

In this study, we discuss through numerical examples the importance of this global constraint on the solution of lake circulation problems and present an algorithm to introduce such a constraint so as to keep the generality and efficiency of a solution strategy in computer code for non-linear problems.

FORMULATION

A finite element model for a non-linear and non-stationary problem leads to a set of algebraic equations:[5]

$$W = \langle \delta \mathbf{U} \rangle \{\mathbf{R}(\mathbf{U})\} = 0,$$

or

$$\{\mathbf{R}(\mathbf{U})\} = \{\mathbf{F}\} - [\mathbf{K}(\mathbf{U})]\{\mathbf{U}\} = \mathbf{0}, \tag{1}$$

where $\{\mathbf{U}\}$ represents the unknowns and $\{\delta\mathbf{U}\}$ the corresponding variations; $\{\mathbf{F}\}$ is the load vector, $\{\mathbf{R}\}$ is the residue vector and $[\mathbf{K}]$ is the coefficient matrix.

Equation (1) is solved by a variant of Newton's method after introducing boundary conditions of the type

$$U_i = \bar{U}_i. \tag{2}$$

A global constraint among certain variables U_i representing water levels, for example, leads to a condition of the type

$$\langle \mathbf{a} \rangle \{\mathbf{U}\} = V_0, \tag{3}$$

where $\langle \mathbf{a} \rangle$ is a vector of non-zero area weighting coefficients for the water-level components of $\{\mathbf{U}\}$ and V_0 is the volume to be conserved.

Equation (3) may be used to define one variable U_p (representing water level) in terms of other variables in order to eliminate it from equation (1). This would introduce undesirable restructuring of the matrix $[\mathbf{K}]$ and is difficult to implement in a standard computer code. In the following sections, we present an algorithm for introducing such a constraint requiring no matrix restructuring. For a better understanding, the presentation starts with a linear problem followed by a non-linear problem.

LINEAR MODEL

Let us consider a problem defined by

$$W = \langle \delta\mathbf{U} \rangle ([\mathbf{K}]\{\mathbf{U}\} - \{\mathbf{F}\}) = 0,$$

or

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}\}. \tag{4}$$

The condition (3) may be introduced in equation (4) directly by defining

$$U_p = \frac{V_0}{a_p} - \langle \bar{\mathbf{a}} \rangle \{\bar{\mathbf{U}}\}, \tag{5a}$$

where $\bar{a}_i = a_i/a_p$ and the pth component is removed:

$$\delta U_p = -\langle \bar{\mathbf{a}} \rangle \{\delta \bar{\mathbf{U}}\}. \tag{5b}$$

The component p is chosen such that $a_p \neq 0$ and is preferably the largest in absolute value. Equations (5) are used to eliminate U_p and δU_p from equation (4), leading to a completely restructured matrix with an undesirably large bandwidth. If all coefficients $\bar{a}_i$ are non-zero, the modified matrix $[\bar{\mathbf{K}}]$ will be full.

Obviously, in practical studies, such a technique for introducing the constraint is to be discarded. We present thus an indirect method which requires no restructuring of the matrix $[\mathbf{K}]$ and is thus efficient from a computational point of view.

One seeks the solution of equation (4) under two separate conditions of U_p: the solution $\{\mathbf{U}_\mathrm{F}\}$ from $U_p = 0$ and the solution $\{\mathbf{U}_\mathrm{H}\}$ from $U_p = 1$. The two solutions are then combined to obtain $\{\mathbf{U}\}$ satisfying equation (3):

$$[\mathbf{K}]\{\mathbf{U}_\mathrm{F}\} = \{\mathbf{F}\}, \tag{6a}$$

$$[\mathbf{K}]\{\mathbf{U}_\mathrm{H}\} = \{\mathbf{H}\}, \tag{6b}$$

with

$$k_{pp} = k_{pp} + A,$$

$$H_i = 0, \quad i \neq p; H_p = k_{pp} + A.$$

Equations (6) are solved using the same triangulatization of $[\mathbf{K}]$ with two loads vectors $\{\mathbf{F}\}$ and $\{\mathbf{H}\}$. There is no matrix restructuring except the modification of the diagonal term k_{pp} by a large value A. This would lead to $U_p \approx 0$ for equation (6a) and to $U_p \simeq 1$ for equation 6(b). The value of A is large as compared to the largest component of the pth line in absolute value:[5]

$$\frac{A}{\max|k_{pi}|} \simeq 10^8.$$

Then

$$\{\mathbf{U}\} = \{\mathbf{U}_\mathrm{F}\} + \lambda\{\mathbf{U}_\mathrm{H}\}, \tag{7}$$

so that

$$\langle \mathbf{a} \rangle (\{\mathbf{U}_\mathrm{F}\} + \lambda\{\mathbf{U}_\mathrm{H}\}) = V_0,$$

leading to

$$\lambda = \frac{V_0 - \langle \mathbf{a} \rangle \{\mathbf{U}_\mathrm{F}\}}{\langle \mathbf{a} \rangle \{\mathbf{U}_\mathrm{H}\}}. \tag{8}$$

The increase in computational effort due to having two load vectors, $\{\mathbf{F}\}$ and $\{\mathbf{H}\}$, is relatively small and this technique may be easily implemented in a computer code.

Remark

We employ the technique of a large diagonal term in order to introduce the constraint on the U_p component. This is primarily due to the general structure of our computer code, which considers the turbulent boundary layer through a diagonal term on the appropriate nodes. However, if one encounters numerical instability, one may explicitly eliminate the pth line and column of $[\mathbf{K}]$ and modify the corresponding $\{\mathbf{F}\}$ and $\{\mathbf{H}\}$ in equation (6) for $U_p = 0$ and $U_p = 1$. In the case studies we have undertaken on VAX-VMS machines with double precision, no numerical instability has been observed with the large diagonal term technique.

NON-LINEAR PROBLEM

The introduction of the constraint (3) follows similar steps to those of the linear case; let

$$\{\mathbf{R}\} = \{\mathbf{F}\} - [\mathbf{K}(\mathbf{U})]\{\mathbf{U}\} = \mathbf{0}. \tag{9}$$

The Newton–Raphson solution method for (9) is

$$\begin{aligned} \{\mathbf{R}^i\} &= \{\mathbf{F}^i\} - [\mathbf{K}(\mathbf{U}^i)\{\mathbf{U}^i\}], \\ [\mathbf{K}_\mathrm{T}^i]\{\Delta\mathbf{U}\} &= \{\mathbf{R}^i\}, \\ \{\mathbf{U}^{i+1}\} &= \{\mathbf{U}^i\} + \alpha\{\Delta\mathbf{U}^i\}, \end{aligned} \tag{10}$$

where $[\mathbf{K}_\mathrm{T}]$ is the tangent matrix, α is the relaxation parameter and i is the iteration level.

The constraint (3) in incremental form becomes

$$\langle\mathbf{a}\rangle\{\Delta\mathbf{U}\} = \frac{(V_0 - \langle\mathbf{a}\rangle\{\mathbf{U}\})}{\alpha} = \frac{\Delta V_0}{\alpha}. \tag{11}$$

The introduction of the constraint (11) in the Newton–Raphson solution method is presented in the following algorithmic form:

1. Iteration i
   (a) compute the residual $\{\mathbf{R}\}$ and ΔV_0
   (b) compute $[\mathbf{K}_\mathrm{T}]$ and modify $k_{\mathrm{T}_{pp}} = k_{\mathrm{T}_{pp}} + A$
   (c) triangularize $[\mathbf{K}_\mathrm{T}]$
   (d) calculate λ

$$\lambda = \frac{\Delta V_0/\alpha - \langle\mathbf{a}\rangle\{\Delta\mathbf{U}_\mathrm{R}\}}{\langle\mathbf{a}\rangle\{\Delta\mathbf{U}_\mathrm{H}\}}$$

   (e) $\{\Delta\mathbf{U}\} = \{\Delta\mathbf{U}_\mathrm{R}\} + \lambda\{\Delta\mathbf{U}_\mathrm{H}\}$
   (f) update the solution: $\{\mathbf{U}\} = \{\mathbf{U}\} + \alpha\{\Delta\mathbf{U}\}$.
2. Convergence test.

NUMERICAL EXAMPLES

In order to observe the importance of the water-level constraint for wind-induced flows, we present an academic and a real-life problem. The fluid flow is represented by three-dimensional Navier–Stokes equations for free surface flows with an approximation of hydrostatic pressure and homogeneous density. The finite element model employs an 18-node prismatic element having two horizontal velocity components associated with each node; the water level is assumed to vary linearly associated with three nodes of an element lying in the surface. The vertical velocity w is assumed constant per element and is obtained from the continuity equation. The details of the model may be found in Reference 4.

We study the wind-induced flow in a closed canal as shown in Figure 1. The results are given by Baines and Knapp.[6] In our preliminary tests, different types of boundary conditions in water levels have been introduced in order to assess the sensitivity of convergence of the non-linear model:

A: exact water level values on both ends of the canal
B: zero value (relative water level) at the node lying in the middle
C: zero value (relative water level) at both side nodes in the middle
D: global volume conservation choosing different points for U_p.

Condition A led to rapid convergence, verifying the overall functioning of the non-linear model. However, for practical situations, such a choice is not available. Condition B was found to be divergent, whereas condition C seems to converge slowly provided that the non-linear influences are introduced slowly (five steps with three iterations each). Again for real situations, it is not easy to locate the position of zero water levels; thereby, condition C has no practical value.

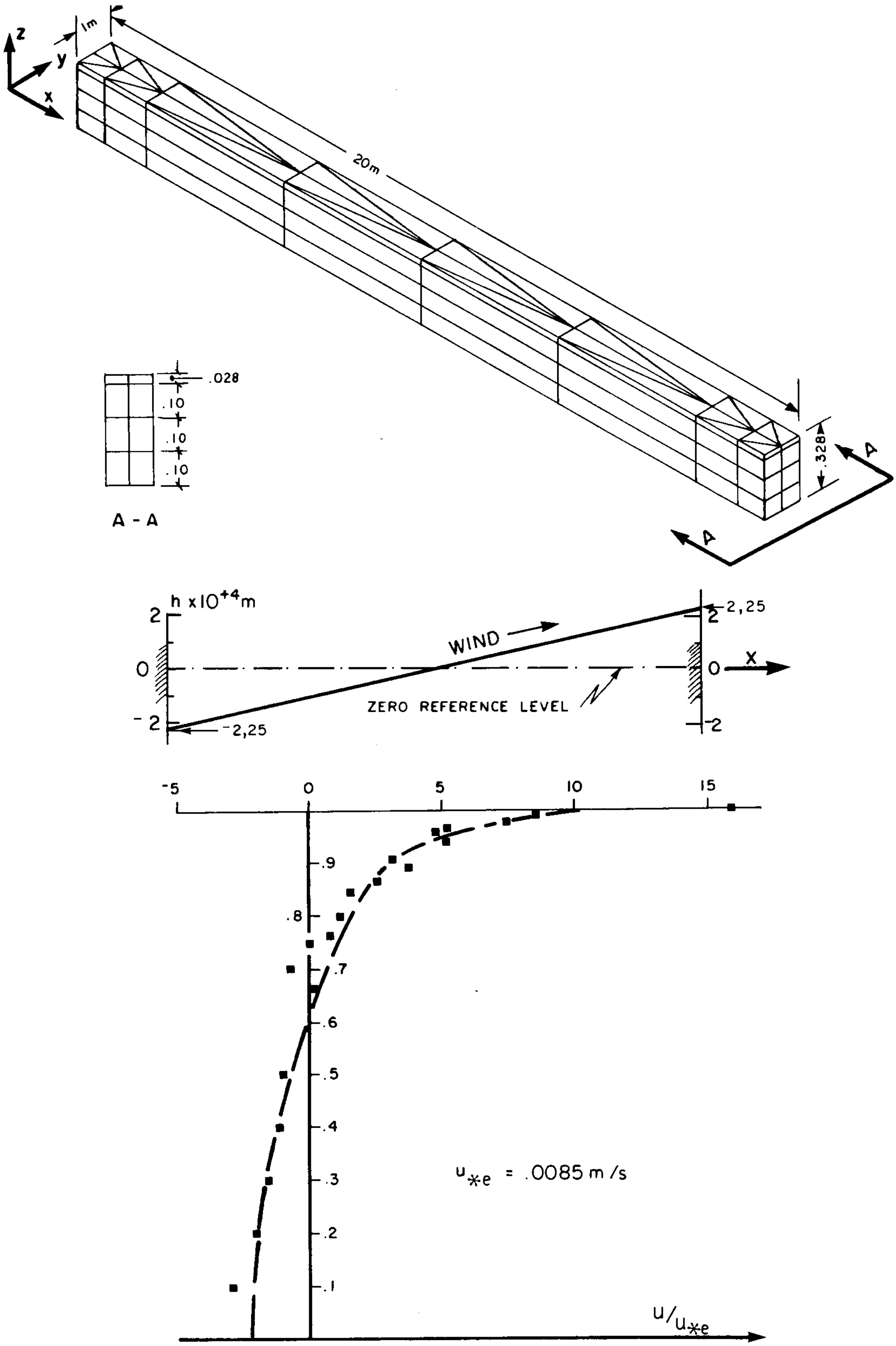

Figure 1. Wind induced circulation in a flume-case study: (a) discretization of the flume;[4] (b) water level results from model;[4] (c) model results for velocity vs. Baines and Knapp's[6]

The introduction of condition D led to a very rapidly convergent solution with one step and three iterations. The constraint U_p is chosen at different nodes and the solution convergence and precision seems to be independant of the choice of node U_p.

Finally, we study the wind-induced circulation in the lake Saint-Jean located in the middle north of the province of Québec in Canada. This problem represents a real-life flow, where it is not possible at all to locate the zero level points. We introduced simply a global volume conservation constraint as discussed above and the convergent solution was obtained in only one step with six iterations. Typical water level contours are shown in Figure 2. The details of the study may be found in Reference 4.

CONCLUSION

It has been shown that wind-induced flows are highly sensitive to choice of water-level reference conditions for practical situations, in which the global constraint of volume conservation is the only available information for introducing water-level conditions.

We have presented an algorithm for introducing a global constraint for a general non-linear problem, which is simple and easy to implement in a general computer code.

Various numerical tests have shown that the convergence characteristics of lake circulation problems are surprisingly improved through introduction of the constraint in the manner presented in this study. Moreover, the technique has proved to be robust and economical for the problems studied so far.

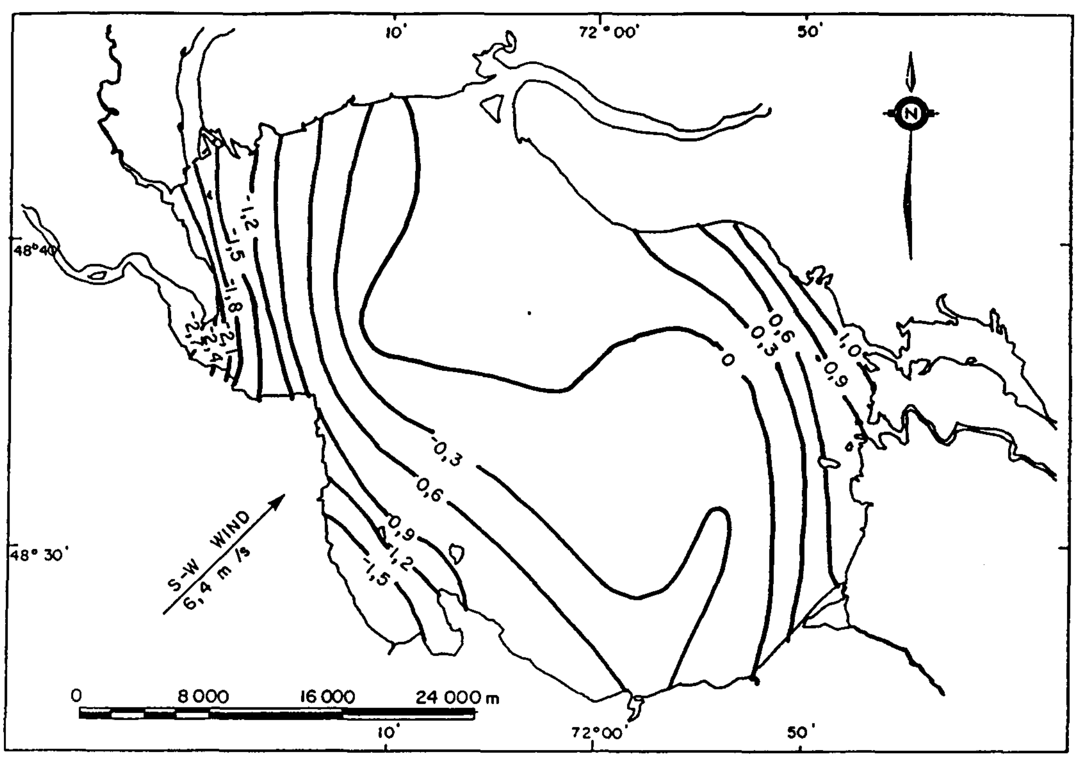

Figure 2. Relative water level contours for a mean south-west wind—lake Saint-Jean, Québec[4]

One may easily employ this technique of introducing global constraints to other situations arising in flow problems, contact problems in elasticity or structural mechanics problems.

REFERENCES

1. C. Taylor and P. Hood, 'Numerical solution of the Navier–Stokes equations using the finite element technique', *Comp. and Fluids*, **1**, (1973).
2. J. J. Connor and C. A. Brebbia, *Introduction to Finite Element Technique for Fluid Flow*, Butterworths, London, 1975.
3. TAO Simulations Inc., 'Simulation des conditions abiotiques du lac Saint-Louis, du bief d'amenée et du Bassin de la Prairie (Archipel de Montréal)', *Rapport scientifique No. TAO-85-03*, 1985.
4. M. Leclerc, 'Modélisation tridimensionnelle des écoulements à surface libre par éléments finis: application au lac Saint-Jean (Québec)', *Thèse de docteur-ingénieur*, UTC, Compiègne (France), 1985.
5. G. Dhatt and G. Touzot, *Une Présentation de la Méthode des éléments Finis*, Maloine, Paris, 1981.
6. W. D. Baines and D. J. Knapp, 'Wind driven water currents', *J. Hyd. Div., ASCE*, **91**(HY2), 205–221 (1965).